

RESIDUES OF A PFAFF SYSTEM RELATIVE TO AN INVARIANT SUBSCHEME

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ABSTRACT. In this paper we give a purely algebraic construction of the theory of residues of a Pfaff system relative to an invariant subscheme. This construction is valid over an arbitrary base scheme of any characteristic.

0. INTRODUCTION

In [2], C. Camacho and P. Sad proved that if \mathfrak{D} is a holomorphic foliation (locally, a vector field) over a complex surface and p is a singular closed point of the foliation, then there exists a holomorphic curve passing through p and the solution of the foliation. Their proof is based in the theorem of reduction of singularities due to Seidenberg and in the introduction of an invariant associated with the singularity and with a curve passing through it and the solution of the foliation. They called this invariant an index. Since then the notion of index has been generalized, with the name of residue, for more general situations. In the case of a singular curve of the projective plane it is due to A. Lins Neto ([7]). If the surface is a complete intersection of $\mathbb{P}^n(\mathbb{C})$ it is due to M. Soares ([9]) and for an arbitrary complex surface to T. Suwa ([10]). In higher dimension, when the subvariety is nonsingular, it has been studied by B. Gmira ([3]) and A. Lins Neto ([8]), in the case of a hypersurface, and the general case is due to D. Lehmann ([6]). Finally, in arbitrary dimension and for a singular subvariety, it is due to D. Lehmann and T. Suwa ([5], [12]). They use the differentiable structure underlying the analytic variety and the existence of connections.

In this paper we give a purely algebraic and far more general construction (valid in arbitrary characteristic) of the theory of residues of a Pfaff system \mathcal{P} (without conditions of integrability), on an arbitrary T -scheme X and relative to a closed subscheme Y which is the solution of \mathcal{P} . The only hypothesis that we make is that the normal module of Y over X be a coherent locally free \mathcal{O}_Y -module. We make a technical hypothesis about the module of differentials $\Omega_{Y/T}$: we suppose that the Koszul complex $\text{Kos}_\bullet(\Omega_{Y/T})$ is acyclic. This is always true in characteristic 0 or when $\Omega_{Y/T}$ is a flat \mathcal{O}_Y -module (in arbitrary characteristic). We think that the hypothesis is unnecessary, that is to say, it is always true (Conjecture 2.3).

The construction is based upon a new construction of the Atiyah extension (Theorems 1.1 and 1.2). This new construction allows one to relate the Atiyah extension

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to the solutions of a Pfaff system (Proposition 4.1), and then to relate the Chern classes to the residues (Theorem 4.2). The local computation of the residue (under fine hypothesis) in terms of the Grothendieck symbol follows easily from the construction (Theorem 4.3). We give special attention to the case of a blowing up for its interest in desingularization. Finally we give an example of computation in the singular case.

1. GENERALIZED LOGARITHMIC DIFFERENTIALS. THE ATIYAH EXTENSION

Let Y be a scheme over an arbitrary scheme T and N a locally free \mathcal{O}_Y -module.

Definition. We denote by $\mathcal{O}_Y[N]$ the \mathcal{O}_Y -algebra

$$\mathcal{O}_Y[N] = \mathcal{O}_Y \oplus N$$

with the product $(a, n) \cdot (a', n') = (aa', an' + a'n)$; Therefore $N^2 = 0$. If we denote \mathcal{I} the irrelevant ideal of $S^\bullet N$, then there is an isomorphism of \mathcal{O}_Y -algebras:

$$\mathcal{O}_Y[N] = (S^\bullet N)/\mathcal{I}^2.$$

Geometrically, if $Y \hookrightarrow \text{Spec } S^\bullet N$ is the null section (which is defined by the ideal \mathcal{I}), then $\text{Spec } \mathcal{O}_Y[N]$ is the first infinitesimal neighbourhood.

Let us denote $X = \text{Spec } \mathcal{O}_Y[N]$ and $\mathcal{O}_X = \mathcal{O}_Y[N]$. One has a natural morphism $\pi: X \rightarrow Y$, which corresponds with the natural morphism of \mathcal{O}_Y -algebras $\mathcal{O}_Y \rightarrow \mathcal{O}_X$, and an immersion $Y \xrightarrow{j} X$, which corresponds with the epimorphism of \mathcal{O}_Y -algebras $\mathcal{O}_X \rightarrow \mathcal{O}_Y$, with kernel N .

For each \mathcal{O}_Y -module M , denote

$$M_X = M \otimes_{\mathcal{O}_Y} \mathcal{O}_X = M \oplus (M \otimes_{\mathcal{O}_Y} N).$$

Let us denote

$$\text{Der}^N(\mathcal{O}_X, M_X)$$

the \mathcal{O}_X -module of the derivations (over T) of \mathcal{O}_X in M_X that leave N stable (that is, that take N into $N \cdot M_X = M \otimes_{\mathcal{O}_Y} N$).

Proposition 1.1. *One has an exact and locally split sequence:*

$$0 \rightarrow \text{Hom}_Y(N, M \otimes_{\mathcal{O}_Y} N) \rightarrow \text{Der}^N(\mathcal{O}_X, M_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \rightarrow \text{Der}(\mathcal{O}_Y, M) \rightarrow 0$$

which is functorial in M .

Proof. If $D: \mathcal{O}_X \rightarrow M_X$ is a derivation that leaves N stable, then it induces a derivation in the quotient $\mathcal{O}_Y = \mathcal{O}_X/N \rightarrow M_X/N = M$. That is, one has a morphism

$$\text{Der}^N(\mathcal{O}_X, M_X) \rightarrow \text{Der}(\mathcal{O}_Y, M)$$

and then a morphism

$$\phi: \text{Der}^N(\mathcal{O}_X, M_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \rightarrow \text{Der}(\mathcal{O}_Y, M).$$

On the other hand, a morphism $f: N \rightarrow M \otimes N$ induces a derivation $D_f: \mathcal{O}_X \rightarrow M_X$, defined by $D_f(a, n) = (0, f(n))$, that leaves N stable. Then we have a morphism

$$\text{Hom}_Y(N, M \otimes_{\mathcal{O}_Y} N) \rightarrow \text{Der}^N(\mathcal{O}_X, M_X)$$

and then a morphism

$$i: \operatorname{Hom}_Y(N, M \otimes_{\mathcal{O}_Y} N) \rightarrow \operatorname{Der}^N(\mathcal{O}_X, M_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y.$$

The exactness and local triviality of the sequence follows if N is a free module and e_1, \dots, e_n is a basis, then we may define a morphism

$$\operatorname{Der}(\mathcal{O}_Y, M) \rightarrow \operatorname{Der}^N(\mathcal{O}_X, M_X)$$

in the following way: given a derivation $D: \mathcal{O}_Y \rightarrow M$, we define $\tilde{D}: \mathcal{O}_X \rightarrow M_X$ by $\tilde{D}(a, \sum a_i e_i) = (Da, \sum (Da_i) \otimes e_i)$, that is, $\tilde{D}e_i = 0$. It is easy to check that \tilde{D} is a derivation that leaves N stable. We have then a morphism

$$s: \operatorname{Der}(\mathcal{O}_Y, M) \rightarrow \operatorname{Der}^N(\mathcal{O}_X, M_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$$

and a straightforward verification shows that the composition $s \circ \phi$ is the identity. Moreover, it is easy to see that if $D \in \operatorname{Der}^N(\mathcal{O}_X, M_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$, then $D - (s \circ \phi)(D)$ is a morphism of N in $M \otimes N$. Conclusion follows. \square

Theorem 1.2 (logarithmic differentials). *The functor*

$$F(M) = \operatorname{Der}^N(\mathcal{O}_X, M_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$$

is representable. The representant will be denoted by $\Omega_N[Y]$ and will be called logarithmic differentials of N . One has an extension

$$0 \rightarrow \Omega_{Y/T} \rightarrow \Omega_N[Y] \rightarrow (\operatorname{End}_Y N)^* \rightarrow 0.$$

Proof. The natural morphism $(\operatorname{End}_Y N) \otimes_{\mathcal{O}_Y} M \rightarrow \operatorname{Hom}_Y(N, M \otimes_{\mathcal{O}_Y} N)$ is an isomorphism, so the functor in M , $\operatorname{Hom}_Y(N, M \otimes_{\mathcal{O}_Y} N)$, is representable by $(\operatorname{End}_Y N)^*$.

On the other hand, the functor $\operatorname{Der}_T(\mathcal{O}_Y, M)$ is representable by the module of differentials, $\Omega_{Y/T}$. One concludes by the Proposition 1.1 before. \square

Remark. If N is the conormal module of a hypersurface Y of a smooth variety X , then $\Omega_N[Y] = \Omega_X[\log Y] \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. That is why we call $\Omega_N[Y]$ logarithmic differentials.

Theorem 1.3. *The extension $0 \rightarrow \Omega_{Y/T} \rightarrow \Omega_N[Y] \rightarrow (\operatorname{End}_Y N)^* \rightarrow 0$ is the Atiyah extension ([1]).*

Proof. It is enough to see that the splittings of the extension (1.2) correspond with the connections over N . Let us denote $T_Y = \operatorname{Der}(\mathcal{O}_Y, \mathcal{O}_Y) = \Omega_{Y/T}^*$, and $\mathcal{O}_X = \mathcal{O}_Y \oplus N$. Given a connection over N , one has a morphism $\nabla: T_Y \rightarrow \operatorname{End}_T(N)$. Then one has a morphism $T_Y \rightarrow \operatorname{Der}^N(\mathcal{O}_X, \mathcal{O}_X)$ defined as $D \mapsto \tilde{D}$, $\tilde{D}(a, n) = (Da, \nabla(D)(n))$, and then a morphism

$$T_Y \rightarrow \operatorname{Der}^N(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y = (\Omega_N[Y])^*$$

and taking dual, $\Omega_N[Y] \rightarrow \Omega_{Y/T}$, which is a section of $\Omega_{Y/T} \rightarrow \Omega_N[Y]$.

Conversely, given a splitting of $0 \rightarrow \Omega_{Y/T} \rightarrow \Omega_N[Y] \rightarrow (\operatorname{End}_Y N)^* \rightarrow 0$, one has a morphism $\Omega_N[Y] \rightarrow \Omega_{Y/T}$, and taking dual $T_Y \rightarrow \operatorname{Der}^N(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. Now,

there is a natural morphism $Der^N(\mathcal{O}_X, \mathcal{O}_X) \rightarrow End_T(N)$ and then a morphism $Der^N(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \rightarrow End_T(N)$. The composition

$$T_Y \rightarrow Der^N(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \rightarrow End_T(N)$$

is the desired connection. \square

2. KOSZUL COMPLEX

The results of this section are well known. They are stated without proofs.

Let \mathcal{O} be a sheaf of rings over a scheme Y , M an \mathcal{O} -module and $S^\bullet M$ its symmetric algebra.

Proposition. *One has an isomorphism of $S^\bullet M$ -modules*

$$\Omega_{S^\bullet M/\mathcal{O}} = M \otimes_{\mathcal{O}} S^\bullet M$$

and therefore $\Omega_{S^\bullet M/\mathcal{O}}^k = \Lambda^k M \otimes_{\mathcal{O}} S^\bullet M$.

The natural morphism $M \rightarrow S^\bullet M$ induces a derivation over \mathcal{O}

$$D: S^\bullet M \rightarrow S^\bullet M$$

which we call *universal derivation of M* . If $S^\bullet M = \mathcal{O}[\xi_1, \dots, \xi_n]$, then $D = \sum_i \xi_i \frac{\partial}{\partial \xi_i}$.

Definition. The inner contraction with D defines a complex

$$\text{Kos}_\bullet(M) \equiv \dots \xrightarrow{i_D} \Omega_{S^\bullet M/\mathcal{O}}^n \xrightarrow{i_D} \Omega_{S^\bullet M/\mathcal{O}}^{n-1} \xrightarrow{i_D} \dots \rightarrow \Omega_{S^\bullet M/\mathcal{O}} \xrightarrow{i_D} S^\bullet M \rightarrow 0$$

which we shall call the *universal Koszul complex of M* . In conclusion,

$$\text{Kos}_\bullet(M) = \Omega_{S^\bullet M/\mathcal{O}}^\bullet = \Lambda^\bullet M \otimes_{\mathcal{O}} S^\bullet M$$

and the differential is the inner contraction with D .

Taking into account that $\Omega_{S^\bullet M/\mathcal{O}}^\bullet = \Lambda^\bullet M \otimes_{\mathcal{O}} S^\bullet M$, there is a bigraduation (p, q) . With regard to it, i_D has degree $(-1, 1)$. If we take the induced simple graduation (that is, $\Lambda^p M \otimes S^q M$ has degree $p + q$), then i_D is homogeneous of degree zero.

Definition. We shall denote $\text{Kos}_\bullet^p(M)$ the homogeneous component of degree p of the complex $\text{Kos}_\bullet(M)$, considering the simple graduation associated with the bigraduation (p, q) . That is, $\text{Kos}_\bullet^p(M)$ is the complex

$$\begin{aligned} 0 \rightarrow \Lambda^p M \xrightarrow{i_D} \Lambda^{p-1} M \otimes M \xrightarrow{i_D} \dots \\ \rightarrow \Lambda^{p-i} M \otimes S^i M \xrightarrow{i_D} \dots \rightarrow M \otimes S^{p-1} M \xrightarrow{i_D} S^p M \rightarrow 0 \end{aligned}$$

and one has that $\text{Kos}_\bullet(M) = \bigoplus_p \text{Kos}_\bullet^p(M)$.

Theorem 2.1. *If \mathcal{O} is an algebra over a field k of characteristic zero, then the complex $\text{Kos}_\bullet(M)$ is acyclic (that is, it is a resolution of \mathcal{O}). In characteristic $p > 0$, if n and p are relatively prime, then $\text{Kos}_\bullet^n(M)$ is acyclic.*

Theorem 2.2. *If M is flat, then the universal Koszul complex of M is acyclic.*

Conjecture 2.3. *The Koszul complex $\text{Kos}_\bullet(M)$ is always acyclic (for an arbitrary module M , in arbitrary characteristic).*

Let $f: M \rightarrow N$ be a morphism of \mathcal{O} -modules, which induces a morphism of \mathcal{O} -algebras $S^\bullet f: S^\bullet M \rightarrow S^\bullet N$.

Definition. We shall call the Koszul complex of f , $\text{Kos}_\bullet(f)$, to the complex

$$\text{Kos}_\bullet(f) = \text{Kos}_\bullet(M) \otimes_{S^\bullet M} S^\bullet N = \Lambda^\bullet M \otimes_{\mathcal{O}} S^\bullet N$$

where the differential is the one induced by the differential of $\text{Kos}_\bullet(M)$. Equivalently, if we denote $D_f: S^\bullet M \rightarrow S^\bullet N$ the derivation which is the composition of $S^\bullet f$ with the universal derivation of M , then $\text{Kos}_\bullet(f)$ is the complex

$$\text{Kos}_\bullet(f) = \Omega_{S^\bullet M/\mathcal{O}}^\bullet \otimes_{S^\bullet M} S^\bullet N$$

and the differential is the inner contraction with D_f .

By definition one has that

$$\text{Kos}_\bullet(\text{Id}) = \text{Kos}_\bullet(M)$$

where $\text{Id}: M \rightarrow M$ is the identity morphism.

Analogously, one defines $\text{Kos}_\bullet^p(f)$ as the homogeneous component of degree p of the complex $\text{Kos}_\bullet(f)$, by considering the simple graduation associated with the bigraduation $\Lambda^p M \otimes S^q N$. That is, $\text{Kos}_\bullet^p(f)$ is the complex

$$\begin{aligned} 0 \rightarrow \Lambda^p M \xrightarrow{i_{D_f}} \Lambda^{p-1} M \otimes N \xrightarrow{i_{D_f}} \\ \dots \rightarrow \Lambda^{p-i} M \otimes S^i N \xrightarrow{i_{D_f}} \dots \rightarrow M \otimes S^{p-1} N \xrightarrow{i_{D_f}} S^p N \rightarrow 0. \end{aligned}$$

Functoriality: Given a commutative diagram of modules and morphism of modules

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ u \downarrow & & \downarrow v \\ \overline{M} & \xrightarrow{\overline{f}} & \overline{N} \end{array}$$

it induces a morphism of complexes $\text{Kos}_\bullet(f) \rightarrow \text{Kos}_\bullet(\overline{f})$. Analogously, for each p , one has a morphism of complexes $\text{Kos}_\bullet^p(f) \rightarrow \text{Kos}_\bullet^p(\overline{f})$.

Proposition 2.4. Let $0 \rightarrow M_1 \xrightarrow{f} M_2 \rightarrow M_3 \rightarrow 0$ be an exact sequence of modules. Suppose that M_3 is flat and that $\text{Kos}_\bullet(M_1)$ is acyclic. Then $\text{Kos}_\bullet(f)$ is a resolution of $S^\bullet M_3$. Analogously $\text{Kos}_\bullet^p(f)$ is a resolution of $S^p M_3$.

If $\text{Kos}_\bullet^p(f)$ is a resolution of $S^p M_3$ of length p , it induces a morphism in cohomology $H^0(Y, S^p M_3) \rightarrow H^p(Y, \Lambda^p M_1)$. Then we have a morphism of graded $H^0(Y, \mathcal{O})$ -algebras

$$(2.5) \quad \theta_f: H^0(Y, S^\bullet M_3) \rightarrow \bigoplus_{p \geq 0} H^p(Y, \Lambda^p M_1)$$

where the second member is an algebra with the cup product associated with the exterior product.

3. CHERN CLASSES

Let us denote $\Omega_{Y/T}^p = \Lambda^p \Omega_{Y/T}$. In this section we are always going to suppose that $\text{Kos}_\bullet(\Omega_{Y/T})$ is acyclic. This is always true in characteristic zero, and in arbitrary characteristic if $\Omega_{Y/T}$ is flat (see Theorems 2.1 and 2.2). This hypothesis is superfluous if Conjecture 2.3 is true.

For simplicity we shall denote $\Omega_Y^p = \Omega_{Y/T}^p$. Let us consider the (Hodge) cohomology ring

$$A^\bullet(Y) = \bigoplus_{p \geq 0} H^p(Y, \Omega_Y^p)$$

with the graduation $A^p(Y) = H^p(Y, \Omega_Y^p)$, and the cup product associated with the exterior product.

Let N be a locally free module over Y . The Atiyah extension $0 \rightarrow \Omega_{Y/T} \rightarrow \Omega_N[Y] \rightarrow (\text{End}_Y N)^* \rightarrow 0$ induces, by the Koszul complex (see (2.5)) a morphism of graded algebras

$$(3.0) \quad \theta_N: H^0(Y, S^\bullet(\text{End}_Y N)^*) \longrightarrow A^\bullet(Y)$$

that is, for each p a morphism $H^0(Y, S^p(\text{End}_Y N)^*) \longrightarrow A^p(Y)$.

Theorem 3.1 ([1, p. 197], [4, pp. 295-296]). *If $s_j \in H^0(Y, S^\bullet(\text{End}_Y N)^*)$ is the function that assigns, to each endomorphism, the j -th coefficient of its characteristic polynomial ($s_1 = -\text{trace}$, \dots , $s_r = (-1)^r \det$), then*

$$\theta_N(s_j) = c_j(N)$$

where $c_j(N)$ is the j -th chern class of N valued at $A^\bullet(Y)$.

4. RESIDUES OF A PFAFF SYSTEM RELATIVE TO AN INVARIANT SUBSCHEME

Let X be a T -scheme. Let us denote Ω_X the module of differentials of X over T .

Definition. A Pfaff system of rank r on X (over T) is a morphism of \mathcal{O}_X -modules

$$\Omega_X \rightarrow \mathcal{P}$$

where \mathcal{P} is a locally free module of rank r . Equivalently, it is a derivation $D: \mathcal{O}_X \rightarrow \mathcal{P}$. Usually we shall simply say that \mathcal{P} is a Pfaff system on X .

If $P = \text{Spec } S^\bullet \mathcal{P}^*$, and $\mathbb{T}_X = \text{Spec } S^\bullet \Omega_X$ is the tangent “bundle”, this is equivalent to giving a morphism

$$P \rightarrow \mathbb{T}_X.$$

In the context of analytic geometry, this would be a holomorphic foliation of dimension r , parametrized by T .

Definition. The *singular locus* of a Pfaff system \mathcal{P} , is the set of points $p \in X$ such that the morphism

$$\Omega_X \otimes k(p) \rightarrow \mathcal{P} \otimes k(p)$$

is not an epimorphism, where $k(p)$ denotes the residual field of p . We shall denote it by $\text{Sin}(\mathcal{P})$.

Equivalently, it is the set of points where the morphism $P \rightarrow \mathbb{T}_X$ is not injective. If C is the cokernel of $\Omega_X \rightarrow \mathcal{P}$, then the singular locus of \mathcal{P} is the support of C . So we shall give it a scheme structure by putting

$$\text{Sin}(\mathcal{P}) = \text{Spec } \mathcal{O}_X / F_0(C)$$

where $F_0(C)$ is the 0-th Fitting ideal of \mathcal{P} .

For each point $x \in X$, let us denote \mathcal{O}_x the local ring at x and $k(x)$ the residual field of x . Let us denote $m_x = \dim_{k(x)} \Omega_{X,x} \otimes_{\mathcal{O}_x} k(x)$ and

$$d(X) = \min_{x \in X} m_x.$$

If X is irreducible, then $d(X) = m_g$, where g is the generic point of X . If $X \rightarrow T$ is smooth, then $d(X)$ is the relative dimension of $X \rightarrow T$.

Definition. We shall denote by $\text{Sin}(X)$ the set of points $x \in X$ such that $m_x > d(X)$. It is a closed subset; indeed, it is the support of $\Lambda^{1+d(X)} \Omega_X$.

Definition. Let \mathcal{P} be a Pfaff system on X and $j: Y \hookrightarrow X$ a closed subscheme. We say that Y is the solution of \mathcal{P} , or that Y is invariant by \mathcal{P} , if the morphism

$$j^* \Omega_X \rightarrow j^* \mathcal{P}$$

factors through Ω_Y . If Y is the solution of \mathcal{P} , then $\mathcal{P}|_Y = j^* \mathcal{P}$ is a Pfaff system on Y .

In terms of the derivation $D: \mathcal{O}_X \rightarrow \mathcal{P}$, Y is the solution of \mathcal{P} if and only if $D\mathfrak{p} \subseteq \mathfrak{p}\mathcal{P}$, where \mathfrak{p} is the sheaf of ideals of Y .

The main fact is the following:

Proposition 4.1. *Let Y be a solution of a Pfaff system \mathcal{P} on X , and let us denote $N = \mathfrak{p}/\mathfrak{p}^2$ the conormal module of Y over X . Suppose that N is a locally free \mathcal{O}_Y -module. Then there exists a morphism*

$$\Omega_N[Y] \rightarrow \mathcal{P}|_Y$$

such that

$$\begin{array}{ccc} \Omega_Y & \xrightarrow{\quad} & \Omega_N[Y] \\ & \searrow & \swarrow \\ & \mathcal{P}|_Y & \end{array}$$

is commutative.

Proof. Let $D: \mathcal{O}_X \rightarrow \mathcal{P}$ be the derivation associated with the Pfaff system. Since Y is the solution of \mathcal{P} , one has that $D\mathfrak{p} \subseteq \mathfrak{p}\mathcal{P}$, so it induces a derivation

$$\mathcal{O}_Y = \mathcal{O}_X / \mathfrak{p} \rightarrow \mathcal{P} / \mathfrak{p}\mathcal{P} = \mathcal{P}|_Y$$

and another

$$\mathfrak{p}/\mathfrak{p}^2 \rightarrow \mathfrak{p}\mathcal{P}/\mathfrak{p}^2\mathcal{P} = \mathfrak{p}/\mathfrak{p}^2 \otimes_{\mathcal{O}_Y} \mathcal{P}|_Y$$

so, taking the direct sum, one has a derivation

$$\mathcal{O}_Y[N] \rightarrow \mathcal{P}|_Y[N]$$

that corresponds with a morphism $\Omega_N[Y] \rightarrow \mathcal{P}|_Y$. The commutativity of the diagram is immediate. \square

Theorem 4.2 (existence of residue). *Let \mathcal{P} be a Pfaff system of rank r on X . Let $Y \xrightarrow{j} X$ be a closed subscheme defined by the sheaf of ideals \mathfrak{p} and solution of \mathcal{P} . Suppose that $\mathfrak{p}/\mathfrak{p}^2$ is a locally free module and denote $S = \text{Sin}(\mathcal{P}|_Y) \cup \text{Sin}(Y)$. If $p > d(Y) - r$, then the morphism (see (3.0))*

$$\theta_p: H^0(Y, S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*) \longrightarrow H^p(Y, \Omega_Y^p)$$

factors through $H_S^p(Y, \Omega_Y^p)$.

Proof. Let us denote $N = \mathfrak{p}/\mathfrak{p}^2$ and

$$\begin{aligned} K' &= \text{kernel of } \Omega_Y \rightarrow \mathcal{P}|_Y, \\ K &= \text{kernel of } \Omega_N[Y] \rightarrow \mathcal{P}|_Y, \\ \overline{K} &= \text{image of } K \text{ in } \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*. \end{aligned}$$

One has a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K' & \xrightarrow{i_{\mathcal{P}}} & K & \longrightarrow & \overline{K} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_Y & \xrightarrow{i} & \Omega_N[Y] & \longrightarrow & \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{P}|_Y & \xlongequal{\quad} & \mathcal{P}|_Y & & \end{array}$$

that induces a morphism of complexes

$$\text{Kos}_{\bullet}(i_{\mathcal{P}}) \rightarrow \text{Kos}_{\bullet}(i).$$

Let $U = Y - S$. Restricting to U , the morphisms $\Omega_Y \rightarrow \mathcal{P}|_Y$ and $\Omega_N[Y] \rightarrow \mathcal{P}|_Y$ are epimorphisms, so K' and K are, over U , locally direct summands of Ω_Y and $\Omega_N[Y]$ respectively, and the quotient \overline{K} is isomorphic to $\text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*$ over U . Therefore $\text{Kos}_{\bullet}(i_{\mathcal{P}})$ is a resolution of $S^{\bullet}\overline{K}$ on U .

Given a sheaf F on Y , let us denote F_U (resp. F_S) the sheaf supported on U (resp. on S). One has an exact sequence $0 \rightarrow F_U \rightarrow F \rightarrow F_S \rightarrow 0$.

By the discussion above, one has that $\text{Kos}_{\bullet}(i_{\mathcal{P}})_U$ is a resolution of the sheaf $(S^{\bullet}\overline{K})_U = (S^{\bullet}\text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*)_U$ and one has an inclusion

$$\text{Kos}_{\bullet}(i_{\mathcal{P}})_U \hookrightarrow \text{Kos}_{\bullet}(i).$$

Let K_{\bullet} be the quotient complex $\text{Kos}_{\bullet}(i)/\text{Kos}_{\bullet}(i_{\mathcal{P}})_U$, which is a resolution of the sheaf $(S^{\bullet}\text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*)_S$. Analogously, for each p , $K_{\bullet}^p = \text{Kos}_{\bullet}^p(i)/\text{Kos}_{\bullet}^p(i_{\mathcal{P}})_U$ is a resolution of $(S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*)_S$. If p is greater than $d(Y) - r$, then $(\Lambda^p K')_U = 0$, because K' is, over U , a locally direct summand of Ω_Y . Therefore, if $p > d(Y) - r$, then K_{\bullet}^p is an extension of $(S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*)_S$ by Ω_Y^p , and the natural morphism

$\text{Kos}_\bullet^p(i) \rightarrow K_\bullet^p$ induces a commutative diagram

$$\begin{array}{ccc} H^0(Y, S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*) & & \\ \downarrow & \searrow & \\ & & H^p(Y, \Omega_Y^p) \\ & \nearrow & \\ H^0(Y, (S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*)_S) & & \end{array}$$

Now,

$$H^0(Y, (S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*)_S) = H_S^0(Y, (S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*)_S)$$

and hence the morphism $H_S^0(Y, (S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*)_S) \rightarrow H^p(Y, \Omega_Y^p)$ factors through $H_S^p(Y, \Omega_Y^p)$, so the conclusion follows. \square

Remark. Let d be the minimum integer such that $\Lambda^d \Omega_Y = 0$. The proof of the theorem shows that if $p \geq d - r$, then the residues are localized not only in $S = \text{Sin}(\mathcal{P}|_Y) \cup \text{Sin}(Y)$ but in $\text{Sin}(\mathcal{P}|_Y)$.

Definition. Let Q be a homogeneous polynomial of l ($= \text{rank of } N_{Y/X}$) variables and degree $p > d(Y) - r$ (assume that the variable x_i has degree i). If S_α are the connected components of $S = \text{Sin}(\mathcal{P}|_Y) \cup \text{Sin}(Y)$, we shall denote by $\text{Res}_{S_\alpha}(\mathcal{P}, Q)$ the image of $Q(s_1, \dots, s_l)$ in $H_{S_\alpha}^p(Y, \Omega_Y^p)$ under the morphism

$$H^0(Y, S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*) \rightarrow H_S^p(Y, \Omega_Y^p) = \bigoplus_{\alpha} H_{S_\alpha}^p(Y, \Omega_Y^p) \rightarrow H_{S_\alpha}^p(Y, \Omega_Y^p).$$

By construction and Theorem 3.1, one has that

$$\sum_{\alpha} \phi_{\alpha}(\text{Res}_{S_\alpha}(\mathcal{P}, Q)) = Q(c_1(\mathfrak{p}/\mathfrak{p}^2), \dots, c_l(\mathfrak{p}/\mathfrak{p}^2))$$

where $\phi_{\alpha}: H_{S_\alpha}^p(Y, \Omega_Y^p) \rightarrow H^p(Y, \Omega_Y^p)$ is the natural morphism and $c_i(\mathfrak{p}/\mathfrak{p}^2)$ is i -th Chern class of $\mathfrak{p}/\mathfrak{p}^2$ with values in the Hodge cohomology.

Remarks. 1) In the complex analytic case the residues are usually expressed as localizations of the characteristic classes of the normal module $\mathfrak{p}/\mathfrak{p}^{2*}$. Taking into account that $c_i(\mathfrak{p}/\mathfrak{p}^2) = (-1)^i c_i(\mathfrak{p}/\mathfrak{p}^{2*})$, one can obtain the definition given in the literature for the complex analytic case from the one given here.

2) The proof of the latter theorem shows why the integrability of the Pfaff system is not necessary. It is only necessary that, out of the singular locus, the kernel of the morphism $\Omega_Y \rightarrow \mathcal{P}|_Y$ is a locally free module of rank $d(Y) - r$.

Explicit computation. *Hypothesis:* Suppose that Y is smooth. Then $S = \text{Sin}(\mathcal{P}|_Y)$. Assume that $S \hookrightarrow Y$ is a locally complete intersection of codimension d . Then

$$H_S^d(Y, \Omega_Y^d) = H^0(S, \mathcal{H}_S^d(Y, \Omega_Y^d))$$

and therefore the residue may be computed locally.

Grothendieck symbol. Let $f_1, \dots, f_d \in H^0(Y, \mathcal{O}_Y)$ be a regular sequence and $w \in H^0(Y, \Omega_Y^d)$. We are going to define an element

$$\begin{bmatrix} \omega \\ f_1, \dots, f_d \end{bmatrix} \in \text{Ext}_{\mathcal{O}_Y}^d(\mathcal{O}_Y/(f_1, \dots, f_d), \Omega_Y^d)$$

which is called the Grothendieck symbol.

Since f_1, \dots, f_d is a regular sequence, the Koszul complex (\mathcal{O} stands for \mathcal{O}_Y)

$$0 \rightarrow \Lambda^d L \rightarrow \dots \rightarrow L \rightarrow \mathcal{O} \rightarrow \mathcal{O}/(f_1, \dots, f_d) \rightarrow 0, \quad L = \mathcal{O} \oplus \dots \oplus \mathcal{O}$$

is acyclic; hence, it gives an element $\text{Kos}_\bullet(f_1, \dots, f_d) \in \text{Ext}_{\mathcal{O}}^d(\mathcal{O}/(f_1, \dots, f_d), \Lambda^d L)$.

Moreover w defines a morphism $\Lambda^d L \rightarrow \Omega_Y^d$, that maps $e_1 \wedge \dots \wedge e_d$ into w , and then a morphism.

$$\text{Ext}_{\mathcal{O}}^d(\mathcal{O}/(f_1, \dots, f_d), \Lambda^d L) \rightarrow \text{Ext}_{\mathcal{O}}^d(\mathcal{O}/(f_1, \dots, f_d), \Omega_Y^d).$$

Then $\begin{bmatrix} \omega \\ f_1, \dots, f_d \end{bmatrix} \in \text{Ext}_{\mathcal{O}_Y}^d(\mathcal{O}_Y/(f_1, \dots, f_d), \Omega_Y^d)$ is the image of $\text{Kos}_\bullet(f_1, \dots, f_d)$ by the latter morphism.

Denote $S = (f_1, \dots, f_d)_0$. Taking into account that $\text{Ext}_{\mathcal{O}}^d(\mathcal{O}/(f_1, \dots, f_d), \Omega_Y^d) \hookrightarrow H_S^d(Y, \Omega_Y^d)$, the Grothendieck symbol $\begin{bmatrix} \omega \\ f_1, \dots, f_d \end{bmatrix}$, considered in $H_S^d(Y, \Omega_Y^d)$, is the image of 1 by the morphisms

$$H^0(S, \mathcal{O}/(f_1, \dots, f_d)) \rightarrow H_S^d(Y, \Lambda^d L) \rightarrow H_S^d(Y, \Omega_Y^d)$$

where $H^0(S, \mathcal{O}/(f_1, \dots, f_d)) \rightarrow H_S^d(Y, \Lambda^d L)$ is the morphism induced by the Koszul complex $0 \rightarrow \Lambda^d L \rightarrow \dots \rightarrow L \rightarrow \mathcal{O} \rightarrow \mathcal{O}/(f_1, \dots, f_d) \rightarrow 0$ and $H_S^d(Y, \Lambda^d L) \rightarrow H_S^d(Y, \Omega_Y^d)$ is the one induced by $\Lambda^d L \rightarrow \Omega_Y^d$.

Local structure of the Pfaff system. Let C be the cokernel of $\Omega_Y \rightarrow \mathcal{P}_Y$. Let us consider the chain of closed subschemes

$$S^{r-1} \subset \dots \subset S^2 \subset S^1 = S$$

defined by $S^i = \{y \in Y \text{ such that } \dim_{k(y)} C_y \otimes k(y) \geq i\}$. S^i is the closed subscheme defined by the Fitting ideal $F_i(C)$. If $y \in S - S^2$, then, by Nakayama's lemma, there exist

$$\begin{aligned} \omega_1, \dots, \omega_n & \text{ basis of } \Omega_{Y,y}, \\ D_1, \dots, D_r & \text{ basis of } \mathcal{P}_{Y,y} \end{aligned}$$

such that the matrix of the morphism $\Omega_{Y,y} \rightarrow \mathcal{P}_{Y,y}$ is

$$\begin{pmatrix} 1 & 0 & & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & f_r & \dots & f_n \end{pmatrix}.$$

That is, $\Omega_{Y,y} \simeq L' \oplus L$ and $\mathcal{P}_{Y,y} \simeq L' \oplus \mathcal{O}$ and the morphism $\Omega_{Y,y} \rightarrow \mathcal{P}_{Y,y}$ is $\text{Id} \oplus (f_r, \dots, f_n)$.

Let e_1, \dots, e_l be a basis of $(\mathfrak{p}/\mathfrak{p}^2)_y = N$. As it was shown in the proof of Proposition 1.1, this basis induces a splitting of the exact sequence $0 \rightarrow \Omega_Y \rightarrow \Omega_N[Y] \rightarrow \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^* \rightarrow 0$; that is, one has a section $\text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^* \rightarrow \Omega_N[Y]$.

Let ϕ_{ij} be the basis of $\text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*$ induced by e_1, \dots, e_l and denote by $\frac{de_i}{e_j}$ their image in $\Omega_N[Y]$ by such a section. Let $g_{ij} \in \mathcal{O}$ be the image of $\frac{de_i}{e_j}$ by the morphism

$$\Omega_N[Y]_y \rightarrow \mathcal{P}_{|Y,y} \simeq L' \oplus \mathcal{O} \rightarrow \mathcal{O}$$

and denote by A the matrix with coefficients g_{ij} .

Theorem 4.3. *With the preceding notations, assume that S has codimension $n - r + 1$ and C is monogenerated at every point. Denote by $c_i(A)$ the i -th coefficient of the characteristic polynomial of A . Then $\text{Res}(\mathcal{P}, Q)$ is locally given by the Grothendieck symbol*

$$\begin{bmatrix} Q(c_1(A), \dots, c_l(A)) \omega_r \wedge \dots \wedge \omega_n \\ f_r \dots f_n \end{bmatrix}$$

if Q is a polynomial of degree $p = n - r + 1$, and it is zero for $p > n - r + 1$.

Proof. First of all, notice that S is locally given by the zeros of f_r, \dots, f_n , and since S has codimension $n - r + 1$, then f_r, \dots, f_n is a regular sequence. Let us consider the diagram (assume $Y = \text{Spec } \mathcal{O}$, with \mathcal{O} a local ring)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_Y & \xrightarrow{i} & \Omega_N[Y] & \longrightarrow & \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^* \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ & & \Omega_Y & \xrightarrow{j} & \mathcal{P}_{|Y} & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & L & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}/(f_r, \dots, f_n) \longrightarrow 0 \end{array}$$

This diagram induces morphisms $\text{Kos}_\bullet^p(i) \rightarrow \text{Kos}_\bullet^p(j) \rightarrow \text{Kos}_\bullet(f_r, \dots, f_n)$, and this latter is a quasi-isomorphism, since $\text{Kos}_\bullet(\text{Id})$ is acyclic. Now, $\text{Kos}_\bullet^p(i)$ is the complex $0 \rightarrow \Omega_Y^p \rightarrow \dots \rightarrow S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^* \rightarrow 0$ which induces a morphism $H^0(Y, S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^*) \rightarrow H_S^p(Y, \Omega_Y^p)$ that takes $Q(s_1, \dots, s_l)$ into $\text{Res}_S(\mathcal{P}, Q)$.

On the other hand, $\text{Kos}_\bullet(f_r, \dots, f_n)$ gives a morphism $H^0(S, \mathcal{O}/(f_r, \dots, f_n)) \rightarrow H_S^p(Y, \Omega_Y^p)$ ($p = n - r + 1$) such that the image of 1 is $\begin{bmatrix} \omega_r \wedge \dots \wedge \omega_n \\ f_r \dots f_n \end{bmatrix}$. Moreover, the morphisms

$$S^p \text{End}_Y(\mathfrak{p}/\mathfrak{p}^2)^* \rightarrow S^p C \rightarrow S^p(\mathcal{O}/(f_r, \dots, f_n)) \simeq \mathcal{O}/(f_r, \dots, f_n)$$

take $Q(s_1, \dots, s_l)$ into $Q(c_1(A), \dots, c_l(A))$. Taking it all together, the result follows. If $p > n - r + 1$, $H^p(\text{Kos}_\bullet(f_r, \dots, f_n)) = 0$ and one concludes. \square

Case of a vector field. Let \mathfrak{D} be a Pfaff system of rank 1, $r = 1$, which we shall call a vector field (in other words, a one-dimensional foliation). As before, $j: Y \hookrightarrow X$ is a closed subscheme which is the solution of \mathfrak{D} and is defined by the sheaf of ideals \mathfrak{p} . We again denote $S = \text{Sin}(\mathfrak{D}_{|Y}) \cap \text{Sin}(Y)$.

Theorem 4.4. *Assume that $\mathfrak{p}/\mathfrak{p}^2$ is a coherent locally free module of rank l . Let S_α be the connected components of S and $Q \in H^0(Y, \mathcal{O}_Y)[s_1, \dots, s_l]$ a homogeneous polynomial of degree $d = \dim Y$. For each S_α there exists an element in $H_{S_\alpha}^d(Y, \Omega_Y^d)$, that we denote by $\text{Res}_{S_\alpha}(\mathfrak{D}, Q)$, such that*

$$(*) \quad \sum_{\alpha} \phi_{\alpha}(\text{Res}_{S_\alpha}(\mathfrak{D}, Q)) = Q(c_1(\mathfrak{p}/\mathfrak{p}^2), \dots, c_l(\mathfrak{p}/\mathfrak{p}^2))$$

where $\phi_\alpha: H_{S_\alpha}^d(Y, \Omega_Y^d) \rightarrow H^d(Y, \Omega_Y^d)$ is the natural morphism.

Remarks. 1) If Y is a proper variety over a field k , one has a canonical morphism

$$H^d(Y, \Omega_Y^d) \rightarrow k$$

which together with the natural morphism $H_{S_\alpha}^d(Y, \Omega_Y^d) \rightarrow H^d(Y, \Omega_Y^d)$ gives a morphism $H_{S_\alpha}^d(Y, \Omega_Y^d) \rightarrow k$. Therefore the residues $\text{Res}_{S_\alpha}(\mathfrak{D}, Q)$ may be considered as numbers (taking their image into k under that morphism). The formula (*) gives us the global sum of these numbers along the subvariety Y .

2) If Y is a variety over a field k and S_α are closed points, one has canonical morphisms

$$H_{S_\alpha}^d(Y, \Omega_Y^d) \rightarrow k$$

and therefore the residues $\text{Res}_{S_\alpha}(\mathfrak{D}, Q)$ may be understood as numbers. If in addition Y is a proper variety, the morphism $H_{S_\alpha}^d(Y, \Omega_Y^d) \rightarrow k$ coincides with the composition of the canonical morphisms $H_{S_\alpha}^d(Y, \Omega_Y^d) \rightarrow H^d(Y, \Omega_Y^d)$ and $H^d(Y, \Omega_Y^d) \xrightarrow{\sim} k$, so the formula (*) gives the global sum of these numbers.

Explicit computation (for a vector field). Let us assume that Y is a smooth variety over a field k and that S_α is a closed point, denoted by p . Let x_1, \dots, x_d be a system of parameters at p and e_1, \dots, e_l a basis of $\mathfrak{p}/\mathfrak{p}^2$ at p . Let D be a local generator of $\mathfrak{D}^* \hookrightarrow T_X$ at p . Let

$$\overline{D} = f_1 \frac{\partial}{\partial x_1} + \dots + f_d \frac{\partial}{\partial x_d}$$

be the local generator of $\mathfrak{D}_{|Y}^* \hookrightarrow T_Y$ at p induced by D , and let L_p be the endomorphism induced in $(\mathfrak{p}/\mathfrak{p}^2)_p$. Denote $c_1(L_p), \dots, c_l(L_p)$ the coefficients of the characteristic polynomial of L_p .

Theorem 4.5. *With the notations above, one has that*

$$\text{Res}_p(\mathfrak{D}, Q) = \text{Res}_p \left(\frac{Q(c_1(L_p), \dots, c_l(L_p)) \cdot dx_1 \wedge \dots \wedge dx_d}{f_1 \dots f_d} \right).$$

(In the right side, Res_p denotes the ordinary residue of a differential form.)

Proof. It follows from 4.3, taking into account that under the canonical morphism $H_p^d(Y, \Omega_Y^d) \rightarrow k$, the Grothendieck symbol

$$\left[\frac{Q(c_1(L_p), \dots, c_l(L_p)) \cdot dx_1 \wedge \dots \wedge dx_d}{f_1 \dots f_d} \right]$$

takes into $\text{Res}_p \left(\frac{Q(c_1(L_p), \dots, c_l(L_p)) \cdot dx_1 \wedge \dots \wedge dx_d}{f_1 \dots f_d} \right)$. \square

Case of a blowing up. Let Z be an irreducible component of codimension d of the singular locus of a vector field \mathfrak{D} on X . Let p be the generic point of Z , $X_p = \text{Spec } \mathcal{O}_p$ and $X' \rightarrow X_p$ the blowing up at p . Let E be the exceptional fibre, which is a projective space of dimension $d-1$ over $k(p) = \text{field of meromorphic functions of } Z$. The vector field in X induces another in X' , that we denote \mathfrak{D}' . Locally this is given as follows:

Let x_1, \dots, x_n be local parameters at a closed nonsingular point of Z , such that $Z = \{x_1 = \dots = x_d = 0\}$. Let

$$f_1 \frac{\partial}{\partial x_1} + \dots + f_n \frac{\partial}{\partial x_n}$$

be a local generator of the vector field. The multiplicity of the vector field at p is the minimum of the multiplicities of the f_i at p . We shall denote it by μ .

Let us take x_1 as a blowing up parameter

$$\begin{aligned} x'_1 &= x_1, \\ x'_i &= \frac{x_i}{x_1}, \quad i = 2, \dots, d, \\ x'_j &= x_j, \quad j = d+1, \dots, n. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial}{\partial x_1} &= \frac{\partial}{\partial x'_1} - \sum_{i=2}^d \frac{x'_i}{x'_1} \frac{\partial}{\partial x'_i}, \\ \frac{\partial}{\partial x_i} &= \frac{1}{x'_1} \frac{\partial}{\partial x'_i}, \quad i = 2, \dots, d, \\ \frac{\partial}{\partial x_j} &= \frac{\partial}{\partial x'_j}, \quad j = d+1, \dots, n. \end{aligned}$$

Then, if we denote

$$f'_i = \frac{f_i}{x_1^\mu}, \quad (\mu = \text{multiplicity of } \mathfrak{D} \text{ at } p)$$

one has that

$$f_1 \frac{\partial}{\partial x_1} + \dots + f_n \frac{\partial}{\partial x_n} = x_1^{\mu-1} \left(f'_1 x'_1 \frac{\partial}{\partial x'_1} + \sum_{i=2}^d (f'_i - x'_i f'_1) \frac{\partial}{\partial x'_i} + \sum_{j=d+1}^n f'_j x'_1 \frac{\partial}{\partial x'_j} \right).$$

Z is said to be nondicritical if $(f'_2 - x'_2 f'_1, \dots, f'_d - x'_d f'_1) \not\subseteq (x'_1)$. Then a local generator of $(\mathfrak{D}')^*$ is

$$\begin{aligned} f'_1 x'_1 \frac{\partial}{\partial x'_1} + \sum_{i=2}^d (f'_i - x'_i f'_1) \frac{\partial}{\partial x'_i} + \sum_{j=d+1}^n f'_j x'_1 \frac{\partial}{\partial x'_j} & \text{ if } Z \text{ is nondicritical,} \\ f'_1 \frac{\partial}{\partial x'_1} + \sum_{i=2}^d \frac{(f'_i - x'_i f'_1)}{x'_1} \frac{\partial}{\partial x'_i} + \sum_{j=d+1}^n f'_j \frac{\partial}{\partial x'_j} & \text{ if } Z \text{ is dicritical,} \end{aligned}$$

which is called *proper transform* of the vector field. It is easy to see that Z is nondicritical if and only if $\mathfrak{D}'|_E^* \hookrightarrow T_{E/k(p)}$.

Suppose that p is a nondicritical singularity of \mathfrak{D} . Then \mathfrak{D}' induces a vector field \mathfrak{D}'' on the normal bundle of E over X' , $\mathbf{N}_{E/X'}$. This vector field is trivial over $k(p)$, that is, it is locally given by a derivation which vanishes over $k(p)$. Moreover E is the solution of \mathfrak{D}'' , considering E as a subvariety of $\mathbf{N}_{E/X'}$ via the null section. In addition, the restriction of \mathfrak{D}'' to E coincides with the restriction of \mathfrak{D}' .

We can apply the residue theory to the Pfaff system \mathfrak{D}'' , taking $\mathbf{N}_{E/X'}$ as the ambient variety and E as the solution subvariety (and considering both of them as varieties over $k(p)$).

All the polynomials of one variable of degree $d-1$ are of the form $\lambda \cdot x^{d-1}$, with $\lambda \in k(p)$. We shall only consider the polynomial $Q = (-x)^{d-1}$. Then Theorems 4.4 and 4.5 stand as follows:

Theorem 4.6. *Let q_i be the generic points of the singular locus of \mathfrak{D}' . For each q_i there exists a “number” in $k(p)$, denoted by $\text{Res}_{q_i}(\mathfrak{D}, E)$, such that*

$$\sum \text{Res}_{q_i}(\mathfrak{D}, E) = c_1(N_{E/X'})^{d-1} = E^d = (-1)^{d-1}$$

via the canonical isomorphism $H^{d-1}(E, \Omega_{E/k(p)}^{d-1}) \xrightarrow{\sim} k(p)$.

Moreover, if q_i has the same codimension as p , then

$$\text{Res}_{q_i}(\mathfrak{D}, E) = \text{Res}_{q_i} \left(\frac{g^{d-1} \cdot dx_2 \wedge \cdots \wedge dx_d}{f_2 \cdots f_d} \right)$$

where g = homothety induced by the vector field in $(N_{E/X'})_{q_i}$ and

$$f_2 \frac{\partial}{\partial x_2} + \cdots + f_d \frac{\partial}{\partial x_d}$$

*is a local generator of $(\mathfrak{D}')^*_{|E} \hookrightarrow T_{E/k(p)}$ at q_i (x_2, \dots, x_d is a local system of parameters at q_i , considering E as a variety over $k(p)$).*

Now we see the relation between the residue and the linear part of the vector field, for singularities of codimension 2.

Let p be a singular point of a vector field \mathfrak{D} , \mathcal{O}_p the local ring at p and \mathfrak{m} the maximal ideal. If D is a local generator of \mathfrak{D}^* at p , it induces an endomorphism

$$L_p: \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2$$

$$\overline{f} \mapsto \overline{Df}$$

which is not zero if and only if \mathfrak{D} has multiplicity 1 at p . We say that L_p is the linear part of the vector field at p .

Corollary 4.7. *Let p be a singularity of codimension 2 and nondicritical and let $X_p = \text{Spec } \mathcal{O}_p$. Let $X' \rightarrow X$ be the blowing up at p , with exceptional fibre E .*

1) $\sum \text{Res}_{q_i}(\mathfrak{D}, E) = -1$. Therefore there exist some q_i such that $\text{Res}_{q_i}(\mathfrak{D}, E) \notin \mathbb{Q}^+ \cup 0$ (in zero characteristic).

2) If $L_{q_i} = \begin{pmatrix} \lambda & \gamma \\ 0 & \mu \end{pmatrix}$ is the linear part of the vector field at q_i , where $\mu \neq 0$ and λ is the eigenvalue corresponding to the exceptional fibre, then

$$\text{Res}_{q_i}(\mathfrak{D}, E) = \text{Tr} \left(\frac{\lambda}{\mu} \right)$$

where $\text{Tr}: k(q_i) \rightarrow k(p)$ is the trace morphism (which is the identity if q_i is rational over p).

3) Let $q \in E$ be a nondicritical singular point and $X'' \rightarrow X'$ the blowing up at q with exceptional fibre E'' . If E' denotes the proper transform of E and $q' = E' \cap E''$, then

$$\text{Res}_{q'}(\mathfrak{D}, E') = \text{Res}_q(\mathfrak{D}, E) - 1.$$

Proof. 1) and 2) follow from Theorem 4.6. 3) follows from Theorem 4.6 and from the equality

$$E^2 = (E' + E'')^2 = E'^2 + 2E' \cdot E'' + E''^2 = E'^2 + 2 - 1 = E'^2 + 1,$$

□

This corollary generalizes the properties of the residue of Camacho and Sad on surfaces for singularities of codimension 2.

A computation in the singular case. Let \mathfrak{D} be a vector field on an integral surface X (possibly singular) and with an invariant curve C (possibly singular too). Assume C is integral and a locally complete intersection in X . Let Σ be the field of functions of C and let us denote $S = \text{Sin}(C) \cap \text{Sin}(\mathfrak{D}|_C)$. Let $p \in S$. Then $\text{Res}_p(\mathfrak{D}, C) \in H_p^1(C, \Omega_C)$. It is easy to see that $H_p^1(C, \Omega_C)$ may be identified with $\Omega_{\Sigma/k} / \Omega_{\mathcal{O}/k}$, where \mathcal{O} is the local ring of C at p . Given $\omega \in \Omega_{\Sigma/k}$, we shall denote by $[\omega]$ its class in $H_p^1(C, \Omega_C)$. One has a morphism $\text{Res}_p: H_p^1(C, \Omega_C) \rightarrow k$, such that $\text{Res}_p([\omega])$ is the ordinary residue at p of the 1-form ω (this is the “ordinary” residue for singular varieties).

Let $p \in S$, and let $f = 0$ be a local (at p) equation of C in S . Let D be a local generator of \mathfrak{D} at p . Then $Df = g \cdot f$. Let $\omega \neq 0$ be a regular 1-form of \mathcal{O} , that is, $\omega \in \Omega_{\mathcal{O}/k}$. Then

Theorem 4.8. *With the preceding notations,*

$$\text{Res}_p(\mathfrak{D}, C) = \left[\frac{g|_C \cdot \omega}{\omega(D|_C)} \right].$$

Then, as a number (via the morphism $\text{Res}_p: H_p^1(C, \Omega_C) \rightarrow k$),

$$\text{Res}_p(\mathfrak{D}, C) = \text{Res}_p \left[\frac{g|_C \cdot \omega}{\omega(D|_C)} \right].$$

This residue may be computed in two ways: 1) In terms of the desingularization of the curve: If $\pi: C' \rightarrow C$ is the desingularization of C , then

$$\text{Res}_p \left[\frac{g|_C \cdot \omega}{\omega(D|_C)} \right] = \sum_{\pi(q)=p} \text{Res}_q \left[\frac{g|_{C'} \cdot \omega}{\omega(D|_{C'})} \right].$$

2) This residue may be expressed as a Grothendieck symbol, and this one may be computed as a Grothendieck symbol on the surface:

$$\text{Res}_p \left[\frac{g|_C \cdot \omega}{\omega(D|_C)} \right] = \left[\frac{g|_C \cdot \omega}{\omega(D|_C)} \right] = \left[\begin{array}{c} g \cdot \omega' \wedge df \\ \omega'(D), f \end{array} \right]$$

where ω' is a regular form in X such that $\omega|_C = \omega$.

Proof. Let us consider the extension of the logarithmic differentials

$$0 \rightarrow \Omega_C \rightarrow \Omega_N[C] \rightarrow \Omega_C \rightarrow 0$$

where N is the conormal module of C over X . If $j: C \hookrightarrow X$ is the natural immersion, then $\Omega_N[C] = j^* \Omega_X[\log C]$. Let \mathcal{O} be the local ring at p , and $\mathcal{E} = \Omega_N[C]_p$. One has that $\mathcal{E} = \Omega_{\mathcal{O}} \oplus \mathcal{O} \cdot \frac{df}{f}$. The vector field \mathfrak{D} gives a morphism $\phi: \mathcal{E} \rightarrow \mathcal{O}$, which maps $\omega + \lambda \frac{df}{f}$ into $\omega(D|_C) + \lambda g|_C$ (with the preceding notations). Let us denote $\overline{\Omega}_{\mathcal{O}}$ the quotient of $\Omega_{\mathcal{O}}$ by its torsion submodule. We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{\mathcal{O}} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O} \longrightarrow 0 \\ & & \downarrow & & \downarrow \phi & & \downarrow \\ 0 & \longrightarrow & \overline{\Omega}_{\mathcal{O}} & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}/I \longrightarrow 0 \end{array}$$

where I is the ideal that defines the singular locus of $\mathfrak{D}|_C$ at p . Then one has a morphism

$$\mathcal{O}/I \rightarrow H_p^1(\overline{\Omega}_{\mathcal{O}}) \xleftarrow{\sim} H_p^1(\Omega_{\mathcal{O}}).$$

By construction, $\text{Res}_p(\mathfrak{D}, C)$ is the image of 1 under this morphism. But, via the identification of $H_p^1(\overline{\Omega}_{\mathcal{O}})$ with $\Omega_{\Sigma}/\Omega_{\mathcal{O}}$, and $H_p^1(\mathcal{O})$ with Σ/\mathcal{O} , one has that \mathcal{O}/I is isomorphic to the kernel of the morphism $\Omega_{\Sigma}/\Omega_{\mathcal{O}} \rightarrow \Sigma/\mathcal{O}$, and the explicit isomorphism is given by the snake lemma applied to the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{\Omega}_{\mathcal{O}} & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}/I \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_{\Sigma} & \longrightarrow & \Sigma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & \Omega_{\Sigma}/\Omega_{\mathcal{O}} & \longrightarrow & \Sigma/\mathcal{O} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

This explicit isomorphism takes 1 into $\left[\frac{g|_C \cdot \omega}{\omega(D|_C)}\right]$, so one concludes. The computations 1) and 2) follow from the general theory of ordinary residues and Grothendieck symbols. \square

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